arXiv:1402.2660v2 [math.FA] 12 Mar 2014

An isometrically universal Banach space with a monotone Schauder basis

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Abstract

We present an isometric version of the complementably universal Banach space $\mathcal{B}$ with a monotone Schauder basis. The space $\mathcal{B}$ is isomorphic to Pełczyński's space with a universal basis as well as to Kadec' complementably universal space with the bounded approximation property.
MSC (2010) Primary: 46B04. Secondary: 46M15, 46M40.
Keywords: Monotone Schauder basis, linear isometry.

1 Introduction

A Banach space X is *complementably universal* for a given class of spaces if every space from the class is isomorphic to a complemented subspace of X. In 1969 Pełczyński [10] constructed a complementably universal Banach space with a Schauder basis. In 1971 Kadec [4] constructed a complementably universal Banach space for the class of spaces with the *bounded approximation property* (BAP). In the same year Pełczyński [8] showed that every Banach space with BAP is complemented in a space with a basis. Pełczyński & Wojtaszczyk [11] constructed in 1971 a universal Banach space for the class of spaces with a finite-dimensional decomposition. Applying Pełczyński's decomposition argument [9], one immediately concludes that all three spaces are isomorphic. It is worth mentioning a negative result of Johnson & Szankowski [3] saying that no separable Banach space can be complementably universal for the class of all separable spaces. The author in [2] presented a natural extension property that describes an isometric version of the Kadec-Pełczyński-Wojtaszczyk space. The constructed space is unique, up to isometry, for the class of Banach spaces with finite-dimensional decomposition and isomorphic to the Kadec-Pełczyński-Wojtaszczyk space.

In this note we present an isometric version of the complementably universal Banach space with a monotone Schauder basis. Most of the arguments are inspired by the recent works [2], [6] and [7].

2 Preliminaries

A *projectional resolution of the identity* (briefly: *PRI*) on a Banach space X is a sequence of norm-one projections $\{P_n\}_{n\in\omega}$ of X satysfying following conditions:

(1) $P_n \circ P_m = P_{min\{n,m\}} = P_m \circ P_n$ for every $n, m \in \omega$;

(2) $\dim(P_n[X]) = n$;

(3) $X = \mathrm{cl} \bigcup_{n\in\omega} P_n[X]$.

A *Schauder basis* is a sequence $\{e_n\}_{n\in\omega}$ of vectors in a Banach space X such that for every $x \in X$ there are uniquely determined scalars $\{a_n\}_{n\in\omega}$ such that

$$x = \sum_{n=0}^{\infty} a_n e_n,$$

where the convergence of the series is taken with respect to the norm. Once this happen, for each $n \in \omega$ there is a cannonical projection P_n defined by

$$P_n(\sum_{i\in\omega} a_i e_i) = \sum_{i<n} a_i e_i.$$

A Banach space X has a *monotone Schauder basis* if and only if it has a PRI $\{P_n\}_{n\in\omega}$. On the other hand, the basis is *monotone* if $\|P_n\| \le 1$ for every $n \in \omega$.
Recall that a Banach space X is *1-complemented* in Y if there exists a projection $P : Y \to X$ such that $\|P\| \le 1$ and $P[Y] = X$.
Given Banach spaces $Y \subseteq X$, we say that Y is an *initial subspace* of X if there is a sequence of norm-one projections $\{P_n\}_{n\in\omega}$ satysfying conditions (1), (3) and

1° for each $n \in \omega$ the image $P_{n+1} - P_n$ is 1-dimensional,

2° $X = P_0[Y]$.

Typical examples of initial subspaces are linear spans of initial parts of a Schauder basis. Note that, an initial subspace is 1-complemented and the trivial space is initial in Y if and only if Y has a monotone Schauder basis.
Given a Schauder basis $\{e_n\}_{n\in\omega}$ in X, given a subset $S \subset \omega$, we say that $\{e_n\}_{n\in S}$ is a *cannonically 1-complemented subbasis* if the linear operator $P_S : X \to X$ defined by conditions $P_S e_n = e_n$ for $n \in S$, $P_S e_n = 0$ for $n \notin S$, has norm ≤ 1.

Finally, we say that a basis $\{v_n\}_{n\in\omega}$ is *isometric* to a subbasis of $\{e_m\}_{m\in\omega}$ if there is an increasing function $\varphi : S \to \omega$ such that the linear operator f defined by equations $f(v_n) = e_{\varphi(n)}$ $(S \subseteq \omega)$ is a linear isometric embedding.

Every finite-dimensional Banach space E is isometric to $\mathbb{R}^n$ with some norm $\|\cdot\|$. We shall say that E is *rational* if $E = \mathbb{R}^n$ with a norm such that its unit ball is a polyhedron spanned by finitely many vectors whose every coordinate is a rational number. Equivalently, X is rational if, up to isometry, $X = \mathbb{R}^n$ with a "maximum norm" $\|\cdot\|$ induced by finitely many functionals $\varphi_0, \ldots, \varphi_{m-1}$ such that $\varphi_i[\mathbb{Q}^n] \subseteq \mathbb{Q}$ for every $i < m$. More precisely,

$$\|x\| = \max_{i<m} |\ \varphi_i(x)\ |$$

for $x \in \mathbb{R}^n$. Typical examples of rational Banach spaces are $\ell_1(n)$ and $\ell_\infty(n)$, the n-dimensional variants of ℓ_1 and ℓ_∞, respectively. On the other hand, for $1 < p < \infty$, $n > 1$, the spaces $\ell_p(n)$ are not rational. Of course, every rational Banach space is polyhedral. An operator $T : \mathbb{R}^n \to \mathbb{R}^m$ is *rational* if $T[\mathbb{Q}^n] \subseteq \mathbb{Q}^m$.

It is clear that there are (up to isometry) only countably many rational Banach spaces and for every $\varepsilon > 0$, every finite-dimensional space has an ε-isometry onto some rational Banach space.

Let X is a set. A *convex hull* is the minimal convex set containing X.

Let $\mathfrak{K}$ be a category and $A, B \in \mathfrak{K}$. By $\mathfrak{K}(A, B)$ we will denote the set of all $\mathfrak{K}$-morphisms from A to B. A *subcategory* of $\mathfrak{K}$ is a category $\mathfrak{L}$ such that each object of $\mathfrak{L}$ is an object of $\mathfrak{K}$ and each arrow of $\mathfrak{L}$ is an arrow of $\mathfrak{K}$.

Category $\mathfrak{L}$ is *cofinal* in $\mathfrak{K}$ if for every $A \in \mathfrak{K}$ there exist an object $B \in \mathfrak{L}$ such that the set $\mathfrak{K}(A, B)$ is nonempty. Let $\mathfrak{K}$ be a category. $\mathfrak{K}$ has the *amalgamation property* if for every objects $A, B, C \in \mathfrak{K}$ and for every morphisms $f \in \mathfrak{K}(A, B)$, $g \in \mathfrak{K}(A, C)$ we can find object $D \in \mathfrak{K}$ and morphisms $f' \in \mathfrak{K}(B, D)$, $g' \in \mathfrak{K}(C, D)$ such that $f' \circ f = g' \circ g$.

Category $\mathfrak{K}$ has the *joint embedding property* if for every objects $A, B \in \mathfrak{K}$ we can find some object $C \in \mathfrak{K}$ such that there exist morphisms $f \in \mathfrak{K}(A, C)$, $g \in \mathfrak{K}(B, C)$.

3 The Amalgamation

Lemma 1. (Amalgamation Lemma) *Let Z, X, Y be finite-dimensional Banach spaces, such that $i : Z \to X$, $j : Z \to Y$ are isometric embeddings and $\{0\}$, $i[Z]$, $j[Z]$ are initial subspaces of Z, X, Y, respectively. Then there exists a finite-dimensional Banach space W, isometric embeddings $i' : X \to W$, $j' : Y \to W$ such that $i'[X]$, $j'[Y]$ are initaial subspaces of W and the following commutes:*

$$\begin{array}{ccc} Y & \xrightarrow{j'} & W \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i'} \\ Z & \xrightarrow{i} & X. \end{array}$$

Proof. Let Z, X, Y be finite-dimensional Banach spaces ($\dim(Z) = N$, $\dim(X) = M$, $\dim(Y) = K$ and $N \leq M, K$), where $\{0\}$ is an initial subspace Z, $i[Z]$ is an initial subspace of X and $j[Z]$ is an initial subspace of Y, with sequences of norm-one projections $\{P_n\}_{n\leq N}$, $\{Q_n\}_{n\leq M-N}$, $\{R_n\}_{n\leq K-N}$, respectively. Observe that $Q_0[X] := i(Z) = Z$ and $R_0[Y] := j(Z) = Z$ (we assume that i, j are inclusions).

We define W as a $(X \oplus Y)/\Delta$, where $\Delta = \{(z, -z) : z \in Z\}$. Given $(x, y) \in X \oplus Y$, define a norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$. Let $i'_X(x) = (x, 0) + \Delta$ and $j'_Y(y) = (0, y) + \Delta$. Then i', j' are isometric embeddings (see [2] or [6]).

We have to show that $i'[X], j'[Y]$ are initial subspaces of W (observe that $\dim(W) = M + K - N$). We will definie sequences of projections $\{S_n\}_{n\leq M}$, $\{T_n\}_{n\leq K}$ such that $S_0 := i' \circ Q_{M-N} = (Q_{M-N}, 0) + \Delta$ and $T_0 := j' \circ R_{K-N} = (0, R_{K-N}) + \Delta$. Note that $Q_0[X] = R_0[Y]$. Define $S_n(x, y) := (Q_{M-N}(x), R_n(y)) + \Delta$ for $n \geq 1$. Similarly, $T_n(x, y) := (Q_n(x), R_{K-N}(y)) + \Delta$ for $n \geq 1$.

It is easy to show that $\|S_n\| \leq 1$ and $\|T_n\| \leq 1$. Observe that:

1. $\mathrm{dist}((Q_{M-N}(x), R_n(y))+\Delta) \leq \|(Q_{M-N}(x), R_n(y))\| = \|Q_{M-N}(x)\|_X + \|R_n(y)\|_Y \leq \|x\|_X + \|y\|_Y$;
2. $\mathrm{dist}((Q_n(x), R_{K-N}(y))+\Delta) \leq \|(Q_n(x), R_{K-N}(y))\| = \|Q_n(x)\|_X + \|R_{K-N}(y)\|_Y \leq \|x\|_X + \|y\|_Y$.

This completes the proof ∎

Fix $\varepsilon > 0$ and fix a linear operator $f : X \to Y$ such that

$$(1 + \varepsilon)^{-1} \cdot \|x\| \leq \|f(x)\| \leq \|x\|$$

for $x \in X$ and $f[X]$ is an initial subspace of Y. Consider the following category $\mathfrak{K}^{\varepsilon}_f$. The objects of $\mathfrak{K}^{\varepsilon}_f$ are pairs (i, j) of linear operators $i : X \to Z$, $j : Y \to Z$ between Banach spaces with initial subspaces such that

1. $i[X]$, $j[Y]$ are initial subspaces of Z;
2. $\|i\| \leq 1$ and $\|j\| \leq 1$;
3. $\|i(x) - j(f(x))\| \leq \varepsilon \cdot \|x\|$ for $x \in X$.

Given $a_0 = (i_0, j_0)$ and $b_0 = (i_1, j_1)$ in $\mathfrak{K}^{\varepsilon}_f$, where $i_k : X \to Z_k$ and $j_k : Y \to Z_k$ for $k < 2$, an arrow from a_0 to b_0 is defined to be a linear operator $T : Z_0 \to Z_1$ such that $\|T\| \leq 1$, $T \circ i_0 = i_1$, and $T \circ j_0 = j_1$.

Lemma 2. *The category $\mathfrak{K}^{\varepsilon}_f$ has an initial object (i_0, j_0) such that both i_0, j_0 are canonical isometric embeddings into $X \oplus Y$ with a suitable norm $\|\cdot\|$ and $X \oplus Y$ has initial subspaces X, Y.*

Proof. Define
$$G = \{(x, -f(x)) \in X \times Y : x \in \varepsilon^{-1} B_X\}.$$
Recall that B_X and B_Y are the unit balls of X and Y respectively. Finally, let K be the convex hull of $G \cup (B_X \times \{0\}) \cup (\{0\} \times B_Y)$ and let

$$\|(x,y)\|_K = \inf\{\|u\|_X + \|v\|_Y + \varepsilon\|w\|_X : (x,y) = (u,0) + (0,v) + (w, -f(w)), (x,y) \in K\}$$

on K. We claim that $\|\cdot\|_K$ is as required.
Define linear operators $i_0(x) = (x, 0)$ and $j_0(y) = (0, y)$. We check that $\|i_0(x) - j_0(f(x))\|_K \leq \varepsilon\|x\|_X$.
We have that $\|(x, -f(x))\|_K \leq \varepsilon\|x\|_X$. This implies that

$$\|i_0(x) - j_0(f(x))\|_K = \|(x, -f(x))\|_K \leq \varepsilon\|x\|_X.$$

This proves that (i_0, j_0) is an object of the category $\mathfrak{K}^{\varepsilon}_f$.
Let $(w, -f(w)) \in G$, $(u, 0) \in B_X \times \{0\}$, $(0, v) \in \{0\} \times B_Y$ and let $\|u\| \leq 1$, $\|v\| \leq 1$, $\|w\| \leq \varepsilon^{-1}$. Then $(x, y) \in K$ is a linear combination:

$$(x, y) = (u, 0) + (0, v) + (w, -f(w)) = (u + w, v - f(w)).$$

Now we prove that $\|(x, 0)\|_K = \|x\|_X$ and $\|(0, y)\|_K = \|y\|_Y$. Suppose that $\|x\| = 1$ and $y = 0$; then $v = f(w)$.
It is easy to show that $\|(x, 0)\|_K \leq \|x\|_X$ (we take $u = x$, then $w = 0$ and $v = f(w) = 0$). Observe that

$$\begin{aligned}&\|u\|_X + \|v\|_Y + \varepsilon\|w\|_X = \|u\|_X + \|f(w)\|_Y + \varepsilon\|w\|_X \geq \\ &\geq \|u\|_X + (1 - \varepsilon)\|w\|_X + \varepsilon\|w\|_X = \|u\|_X + \|w\|_X \geq \|x\|_X.\end{aligned}$$

Suppose that $x = 0$ and $\|y\| = 1$, then $u = -w$. It is obvious that $\|(0, y)\|_K \leq \|y\|_Y$ (we take $v = y$, then $f(w) = 0$ and $u = -w = 0$). On the other hand

$$\begin{aligned}&\|u\|_X + \|v\|_Y + \varepsilon\|w\|_X = \|w\|_X + \|v\|_Y + \varepsilon\|w\|_X = \\ &= (1 + \varepsilon)\|w\|_X + \|v\|_Y \geq (1 + \varepsilon)\|w\|_X + \|v\|_Y \geq \|f(w)\|_Y + \|v\|_Y \geq \|y\|_Y.\end{aligned}$$

This proves that i_0, j_0 are isometric embeddings.

We have to check that the convex hull K is a unit ball of the norm $\|\cdot\|_K$. Let

$$B_K = \{(x, y) : \|(x, y)\|_K \leq 1\}.$$

The inclusion $K \subseteq B_K$ is obvious. To prove that $B_K \subseteq K$, fix (x, y) such that $\|(x, y)\|_K < 1$. Then
$$\|u\|_X + \|v\|_Y + \varepsilon\|w\|_X < 1$$

for some u, v, w such that $x = u + w$ and $y = v - f(w)$. Observe that $(u, 0) \in B_X \times \{0\}$, $(0, v) \in \{0\} \times B_Y$, $(w, -f(w)) \in G$ and $(u, 0) + (0, v) + (w, -f(w)) = (x, y)$.

Denote $\alpha = \|u\|_X + \|v\|_Y + \varepsilon\|w\|_X$, then $\alpha < 1$. Let $\lambda_1 = \frac{\|u\|_X}{\alpha}$, $\lambda_2 = \frac{\|v\|_Y}{\alpha}$ and $\lambda_3 = \frac{\varepsilon\|w\|_X}{\alpha}$. Then $(u, 0) = \lambda_1(\frac{\alpha}{\|u\|_X}(u, 0)) \in B_X \times \{0\}$, $(0, v) = \lambda_2(\frac{\alpha}{\|v\|_Y}(0, v)) \in \{0\} \times B_Y$ and $(w, -f(w)) = \lambda_3(\frac{\alpha}{\varepsilon\|w\|_X}(w, -f(w))) \in G$.

We have to check that the pair (i_0, j_0) is an initial object. This means that for every $(i, j) \in Obj(\mathfrak{K}_f^\varepsilon)$ there exists a unique linear operator $T : X \oplus Y \to Z$ such that $T \circ i_0 = i$, $T \circ j_0 = j$ and the norm of T is less or equal to 1.

Fix $(i, j) \in Obj(\mathfrak{K}_f^\varepsilon)$ and define $T(x, y) = i(x) + j(y)$. It is clear that this is the only possibility for T. We will check that $\|T\|_Z \leq 1$.

Let $(x, y) \in K$, then

1° if $(x, y) \in B_X \times \{0\}$ then $\|T(x, 0)\|_Z = \|i(x)\|_Z \leq 1$,

2° if $(x, y) \in \{0\} \times B_Y$ then $\|T(0, y)\|_Z = \|j(y)\|_Z \leq 1$,

3° if $(x, y) \in G$ then $\|T(x, -f(x))\|_Z = \|i(x) - j(f(x))\|_Z \leq \varepsilon \cdot \|x\|_Z \leq \varepsilon \cdot \varepsilon^{-1} = 1$.

Let X and Y be finite-dimensional Banach spaces ($\dim(X) = M$, $\dim(Y) = K$ and $M \leq K$), where $\{0\}$ is an initial subspace X and $f[X]$ is an initial subspace of Y, with sequences of norm-one projections $\{P_n\}_{n \leq M}$ and $\{Q_n\}_{n \leq K-M}$, respectively. Observe that $Q_0[Y] := f(X)$.

We prove that X, Y are initial subspaces of $X \oplus Y$ ($\dim(X \oplus Y) = M + K$). We define sequences of projecions $\{R_n\}_{n \leq K}$, $\{S_n\}_{n \leq M}$ such that $R_0 := i \circ P_M$ and $S_0 := j \circ Q_{K-M}$. Let $R_n := i \circ P_M + j \circ f \circ P_n = (P_M, f \circ P_n)$ for $n \leq M$ and $R_{M+1+n} := i \circ P_M + j \circ Q_{n+1} = (P_M, Q_{n+1})$ for $n \leq K - M - 1$. Similarly $S_n := i \circ P_n + j \circ Q_{K-M} = (P_n, Q_{K-M})$ for $n \leq M$.

It is easy to show that $\|R_n\|_K \leq 1$ and $\|S_n\|_K \leq 1$. Observe that if

1° $\|(P_M, f \circ P_n)\| \leq 1$ for $n \leq M$, we take $u = P_M$, $v = 0$, $w = 0$, $f(w) = f \circ P_n$.

2° $\|(P_M, Q_{n+1})\| \leq 1$ for $n \leq K - M - 1$, we take $u = P_M$, $v = 0$, $w = 0$, $f(w) = Q_{n+1}$.

3° $\|(P_n, Q_{K-M})\| \leq 1$ for $n \leq M$, we take $u = P_n$, $v = 0$, $w = 0$, $f(w) = Q_{K-M}$.

$\square$

First version of the proof of the lemma about extending ε-isometry between Banach spaces can be found in [2] and [7].

4 A construction

In order to make some statements shorter, we shall consider 1-bounded operators, this means linear operators of norm at most 1 only.

We shall now prepare the setup for our construction.

We now define the relevant category $\mathfrak{K}$. The objects of $\mathfrak{K}$ are finite-dimensional Banach spaces. Given finite-dimensional spaces X, Y, an $\mathfrak{K}$-arrow is an isometric embedding $f : X \to Y$ such that $f[X]$ is an initial object of Y and $\{0\}$ is an initial object of X (both have a monotone Schauder basis).

Denote by $\mathfrak{L}$ the subcategory of $\mathfrak{K}$ consisting of all rational $\mathfrak{K}$-arrows. Obviously, $\mathfrak{L}$ is countable. Looking at the proof of Lemma 1, we can see that $\mathfrak{L}$ has the amalgamation property. We now use the concepts from [6] for constructing a "generic" sequence in $\mathfrak{L}$. First of all, a *sequence* in a fixed category $\mathfrak{C}$ is formally a covariant functor from the set of natural numbers ω into $\mathfrak{C}$.

Up to isomorphism, every sequence in $\mathfrak{L}$ corresponds to a chain $\{X_n\}_{n\in\omega}$ of finite-dimensional subspaces with initial subspaces. By this way, the monotone Schauder basis of a Banach space X is translated into the existence of a sequence in $\mathfrak{K}$ whose co-limit is X. For the sake of convenience, we shall denote a sequence by $\vec{U}$, having in mind a chain $\{U_n\}_{n\in\omega}$ of finite-dimensional spaces with the initial subspaces. Given $U_n \subseteq U_m$, the $\mathfrak{K}$-arrow $f_n^m : U_n \to U_m$ is such that the image $f_n^m[U_n]$ is an initial subspace of U_m.

Following [6], we shall say that a sequence $\vec{Y}$ in $\mathfrak{L}$ is *Fraïssé* if it satisfies the following condition:

(A) Given $n \in \omega$, and an $\mathfrak{L}$-arrow $f : Y_n \to Z$, there exist $m > n$ and an $\mathfrak{L}$-arrow $g : Z \to Y_m$ such that $g \circ f$ is the arrow from Y_n to Y_m.

It is clear that this definition is purely category-theoretic. The name "Fraïssé sequence", as in [6], is motivated by the model-theoretic theory of Fraïssé limits explored by Roland Fraïssé [1]. One of the results in [6] is that every countable category with amalgamations has a Fraïssé sequence.

Theorem 1 ([6])**.** *The category $\mathfrak{L}$ has a Fraïssé sequence.*

From now on, we fix a Fraïssé sequence $\{Y_n\}_{n\in\omega}$ in $\mathfrak{L}$. As usual, we assume that the embeddings are inclusions. Let $\mathcal{B}$ be the completion of the union $\bigcup_{n\in\omega} Y_n$.

5 Universality

Theorem 2. *Let X be a Banach space with a monotone Schauder basis. Then there exists an isometric embedding $e : X \to \mathcal{B}$ such that $e[X]$ is an initial subspace of $\mathcal{B}$.*

Proof. Fix a Banach space X with a monotone Schauder basis and let this be witnessed by a chain $\{X_n\}_{n\in\omega}$ together with suitable projections $\{Q_n\}_{n\in\omega}$.

We construct inductively 1-bounded operator $e_n : X_n \to Y_{k_n}$ such that

(1) $e_n[X_n]$ is an initial subspace of Y_{k_n},

(2) $\|e_{n+1} \restriction X_n - e_n\| < 2^{-n}$.

Recall that, according to our previous agreement, we consider only 1-bounded operators. We may assume that $X_0 = Y_0 = \{0\}$, therefore it is clear how to start the induction. Suppose e_n (and $k_n \in \omega$) have already been defined. By Lemma 2, there exist $i : X_n \to W$ and $j : Y_{k_n} \to W$, where $W = X_n \oplus_{e_n} Y_{k_n}$, and the following conditions are satisfied:

(3) $\|j \circ e_n - i\| < 2^{-n}$.

Using Lemma 1, we may further extend W so that there exists also a $\ell : X_{n+1} \to W$ satisfying

(4) $\ell \restriction X_n = i$.

Applying Lemma 2 and Lemma 1 we preserve condition (1). Recall that Y_n is a rational Banach space. Thus, we can extend W further, so that the extended arrow from Y_n to W will become rational. Doing this, we make some "error" of course, although we can still preserve (3) and (4), because all the inequalities appearing there are strict. Now we use the fact that $\{Y_n\}_{n\in\omega}$ is a Fraïssé sequence. Specifically, we find $k_{n+1} > k_n$, rational operators $g : W \to Y_k$ and $H : Y_{k_{n+1}} \to W$ such that $H \circ g = \mathrm{id}_W$ and $g \circ j$ is the inclusion $Y_{k_n} \subseteq Y_k$.

Define $e_{n+1} = g \circ \ell$. This finishes the inductive construction.

Passing to the limits, we obtain 1-bounded operator $e : X \to \mathcal{B}$. Condition (2) imply that e is an isometric embedding. In particular, $e[X]$ is an initial subspace of $\mathcal{B}$.

□

Corollary 3. *The space $\mathcal{B}$ is isomorphic to Pełczyńki's complementably universal space for Schauder bases, as well as to Kadec's complementably universal space for the bounded approximation property.*

Proof. See the proof in [2, Corollary 5.2]. □

6 Isometric uniqueness

Proofs are done analogous like in [2]. We present only the theorems and references, when we can find the proofs.

Let us consider the following extension property of a Banach space E:

(B) Given a pair $X \subseteq Y$ of finite-dimensional Banach spaces with monotone Schauder basis such that $\{0\}$, X are initial subspaces of those spaces, respectively, given an isometric embedding $f : X \to E$ such that $f[X]$ is an initial subspace of E, for every $\varepsilon > 0$ there exists an ε-isometric embedding $g : Y \to E$ such that $\|g \restriction X - f\| < \varepsilon$ and $g[Y]$ is an initial subspace of E.

Theorem 4. *$\mathcal{B}$ satisfies condition (B).*

Proof. See the proof in [2, Theorem 6.1]. □

Lemma 3. *Assume X satisfies condition (B). Then, given $\varepsilon, \delta > 0$, given finite-dimensional spaces $E \subseteq F$, given an ε-isometric embedding $f : E \to X$ such that $f[E]$ is an initial subspace of X, there exists a δ-isometric embedding $g : F \to X$ such that $\|g \restriction E - f\| < \varepsilon$ and $g[F]$ is an initial subspace of X.*

Proof. See the proof in [2, Lemma 6.2]. □

Theorem 5. *Let $\mathcal{B}$ and $\mathcal{K}$ be Banach spaces with monotone Schauder bases satisfying condition (B) and let $h : A \to B$ be a bijective linear isometry between finite-dimensonal subspaces, where A and B are initial subspaces of $\mathcal{B}$ and $\mathcal{K}$, respectively. Then for every $\varepsilon > 0$ there exists a bijective linear isometry $H : \mathcal{B} \to \mathcal{K}$ that is ε-close to h. In particular, $\mathcal{B}$ and $\mathcal{K}$ are linearly isometric.*

Proof. See the proof in [2, Theorem 6.3]. We use standard back-and-forth argument. Instead of codition (4) from proof of [2, Theorem 6.3], we have that $f_n[A_n]$ and $g_n[B_n]$ are initial subspaces of $\mathcal{K}$ and $\mathcal{B}$, respectively. □

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